Lecture 2

Maxwell's Equations in Differential Operator Form

2.1   Gauss's Divergence Theorem

The divergence theorem is one of the most important theorems in vector calculus [29, 31–33] First, we will need to prove Gauss's divergence theorem, namely, that:

$$\iiint_V dV \nabla \cdot \mathbf{D} = \oiint_S \mathbf{D} \cdot d\mathbf{S} \tag{2.1.1}$$

In the above, $\nabla \cdot \mathbf{D}$ is defined as

$$\nabla \cdot \mathbf{D} = \lim_{\Delta V \to 0} \frac{\oiint_{\Delta S} \mathbf{D} \cdot d\mathbf{S}}{\Delta V} \tag{2.1.2}$$

and eventually, we will find an expression for it. We know that if $\Delta V \approx 0$ or small, then the above,

$$\Delta V \nabla \cdot \mathbf{D} \approx \oiint_{\Delta S} \mathbf{D} \cdot d\mathbf{S} \tag{2.1.3}$$

First, we assume that a volume V has been discretized[1] into a sum of small cuboids, where the i-th cuboid has a volume of ΔV_i as shown in Figure 2.1. Then

$$V \approx \sum_{i=1}^{N} \Delta V_i \tag{2.1.4}$$

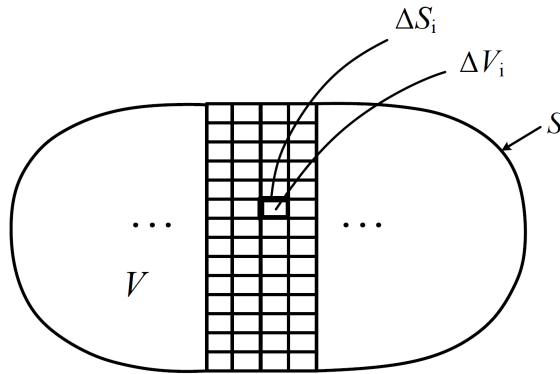[1]Other terms are "tesselated", "meshed", or "gridded".

Figure 2.1: The discretization of a volume V into sum of small volumes ΔV_i each of which is a small cuboid. Stair-casing error occurs near the boundary of the volume V but the error diminishes as $\Delta V_i \to 0$.
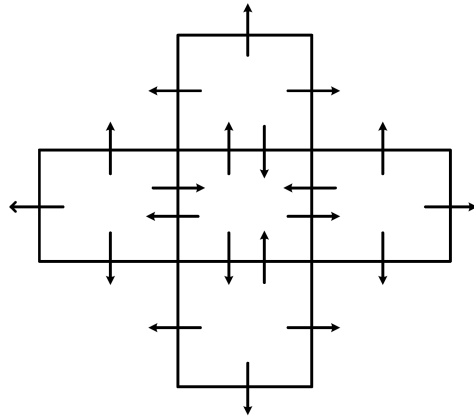

Figure 2.2: Fluxes from adjacent cuboids cancel each other leaving only the fluxes at the boundary that remain uncancelled. Please imagine that there is a third dimension of the cuboids in this picture where it comes out of the paper.

Then from (2.1.2),

$$\Delta V_i \nabla \cdot \mathbf{D}_i \approx \oiint_{\Delta S_i} \mathbf{D}_i \cdot d\mathbf{S}_i \tag{2.1.5}$$

By summing the above over all the cuboids, or over i, one gets

$$\sum_i \Delta V_i \nabla \cdot \mathbf{D}_i \approx \sum_i \oiint_{\Delta S_i} \mathbf{D}_i \cdot d\mathbf{S}_i \approx \oiint_S \mathbf{D} \cdot d\mathbf{S} \tag{2.1.6}$$

It is easily seen the the fluxes out of the inner surfaces of the cuboids cancel each other, leaving only the fluxes flowing out of the cuboids at the edge of the volume V as explained in Figure 2.2. The right-hand side of the above equation (2.1.6) becomes a surface integral over the surface S except for the stair-casing approximation (see Figure 2.1). Moreover, this approximation becomes increasingly good as $\Delta V_i \to 0$, or that the left-hand side becomes a volume integral, and we have

$$\iiint_V dV \nabla \cdot \mathbf{D} = \oiint_S \mathbf{D} \cdot d\mathbf{S} \tag{2.1.7}$$

The above is Gauss's divergence theorem.

Next, we will derive the details of the definition embodied in (2.1.2). To this end, we evaluate the numerator of the right-hand side carefully, in accordance to Figure 2.3.
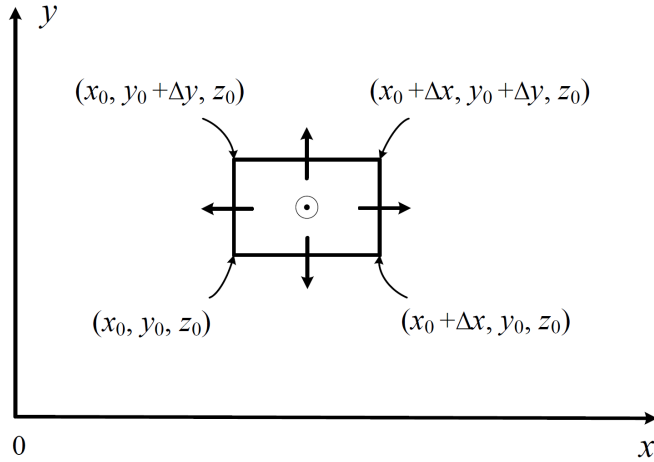

Figure 2.3: Figure to illustrate the calculation of fluxes from a small cuboid where a corner of the cuboid is located at (x_0, y_0, z_0). There is a third z dimension of the cuboid not shown, and coming out of the paper. Hence, this cuboid, unlike as shown in the figure, has six faces.

Accounting for the fluxes going through all the six faces, assigning the appropriate signs in accordance with the fluxes leaving and entering the cuboid, one arrives at

$$\oiint_{\Delta S} \mathbf{D} \cdot d\mathbf{S} \approx -\mathrm{D}_x(x_0, y_0, z_0)\Delta y \Delta z + \mathrm{D}_x(x_0 + \Delta x, y_0, z_0)\Delta y \Delta z$$
$$-\mathrm{D}_y(x_0, y_0, z_0)\Delta x \Delta z + \mathrm{D}_y(x_0, y_0 + \Delta y, z_0)\Delta x \Delta z$$
$$-\mathrm{D}_z(x_0, y_0, z_0)\Delta x \Delta y + \mathrm{D}_z(x_0, y_0, z_0 + \Delta z)\Delta x \Delta y \tag{2.1.8}$$

Factoring out the volume of the cuboid $\Delta V = \Delta x \Delta y \Delta z$ in the above, one gets

$$
\begin{aligned}
\oiint_{\Delta S} \mathbf{D} \cdot d\mathbf{S} \approx \Delta V \, \{ & [D_x(x_0 + \Delta x, \ldots) - D_x(x_0, \ldots)] / \Delta x \\
& + [D_y(\ldots, y_0 + \Delta y, \ldots) - D_y(\ldots, y_0, \ldots)] / \Delta y \\
& + [D_z(\ldots, z_0 + \Delta z) - D_z(\ldots, z_0)] / \Delta z \}
\end{aligned}
\tag{2.1.9}
$$

Or that

$$
\frac{\oiint \mathbf{D} \cdot d\mathbf{S}}{\Delta V} \approx \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}
\tag{2.1.10}
$$

In the limit when $\Delta V \to 0$, then

$$
\lim_{\Delta V \to 0} \frac{\oiint \mathbf{D} \cdot d\mathbf{S}}{\Delta V} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \nabla \cdot \mathbf{D}
\tag{2.1.11}
$$

where

$$
\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}
\tag{2.1.12}
$$

$$
\mathbf{D} = \hat{x} D_x + \hat{y} D_y + \hat{z} D_z
\tag{2.1.13}
$$

The divergence operator $\nabla \cdot$ has its complicated representations in cylindrical and spherical coordinates, a subject that we would not delve into in this course. But they are best looked up at the back of some textbooks on electromagnetics.

Consequently, one gets Gauss's divergence theorem given by

$$
\iiint_V dV \nabla \cdot \mathbf{D} = \oiint_S \mathbf{D} \cdot d\mathbf{S}
\tag{2.1.14}
$$

2.1.1   Gauss's Law in Differential Operator Form

By further using Gauss's or Coulomb's law implies that

$$
\oiint_S \mathbf{D} \cdot d\mathbf{S} = Q = \iiint dV \varrho
\tag{2.1.15}
$$

which is equivalent to

$$
\iiint_V dV \nabla \cdot \mathbf{D} = \iiint_V dV \varrho
\tag{2.1.16}
$$

When $V \to 0$, we arrive at the pointwise relationship, a relationship at a point in space:

$$
\nabla \cdot \mathbf{D} = \varrho
\tag{2.1.17}
$$

2.1.2 Physical Meaning of Divergence Operator

The physical meaning of divergence is that if $\nabla \cdot \mathbf{D} \neq 0$ at a point in space, it implies that there are fluxes oozing or exuding from that point in space [34]. On the other hand, if $\nabla \cdot \mathbf{D} = 0$, if implies no flux oozing out from that point in space. In other words, whatever flux that goes into the point must come out of it. The flux is termed divergence free. Thus, $\nabla \cdot \mathbf{D}$ is a measure of how much sources or sinks exists for the flux at a point. The sum of these sources or sinks gives the amount of flux leaving or entering the surface that surrounds the sources or sinks.

Moreover, if one were to integrate a divergence-free flux over a volume V, and invoking Gauss's divergence theorem, one gets

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = 0 \tag{2.1.18}$$

In such a scenerio, whatever flux that enters the surface S must leave it. In other words, what comes in must go out of the volume V, or that flux is conserved. This is true of incompressible fluid flow, electric flux flow in a source free region, as well as magnetic flux flow, where the flux is conserved.
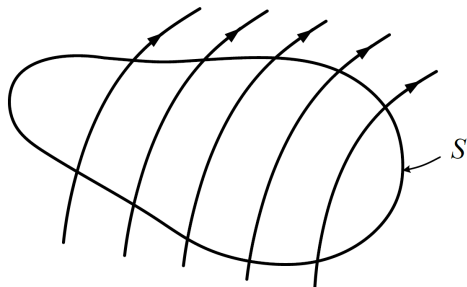
$$\nabla \cdot \mathbf{D} = 0 \quad \Rightarrow \quad \oint_S \hat{n} \cdot \mathbf{D} dS = 0$$

Figure 2.4: In an incompressible flux flow, flux is conserved: whatever flux that enters a volume V must leave the volume V.

2.1.3 Example

If $\mathbf{D} = (2y^2 + z)\hat{x} + 4xy\hat{y} + x\hat{z}$, find:

1. Volume charge density ρ_v at $(-1, 0, 3)$.

2. Electric flux through the cube defined by

$$0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1.$$

3. Total charge enclosed by the cube.

2.2   Stokes's Theorem

The mathematical description of fluid flow was well established before the establishment of electromagnetic theory [35]. Hence, much mathematical description of electromagnetic theory uses the language of fluid. In mathematical notations, Stokes's theorem is

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \iint_S \nabla \times \mathbf{E} \cdot d\mathbf{S} \tag{2.2.1}$$

In the above, the contour C is a closed contour, whereas the surface S is not closed.[2]

First, applying Stokes's theorem to a small surface ΔS, we define a curl operator [3] $\nabla\times$ at a point to be

$$\nabla \times \mathbf{E} \cdot \hat{n} = \lim_{\Delta S \to 0} \frac{\oint_{\Delta C} \mathbf{E} \cdot d\mathbf{l}}{\Delta S} \tag{2.2.2}$$
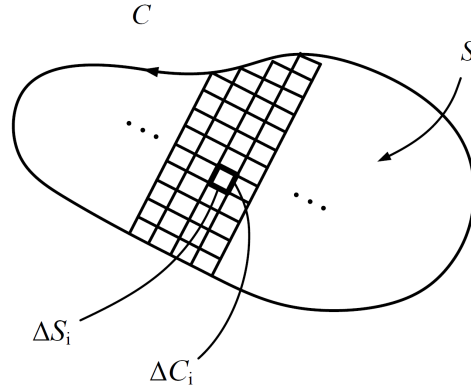

Figure 2.5: In proving Stokes's theorem, a closed contour C is assumed to enclose an open surface S. Then the surface S is tessellated into sum of small rects as shown. Stair-casing error vanishes in the limit when the rects are made vanishingly small.

First, the surface S enclosed by C is tessellated into sum of small rects (rectangles). Stokes's theorem is then applied to one of these small rects to arrive at

$$\oint_{\Delta C_i} \mathbf{E}_i \cdot d\mathbf{l}_i = (\nabla \times \mathbf{E}_i) \cdot \Delta \mathbf{S}_i \tag{2.2.3}$$

---

[2]In other words, C has no boundary whereas S has boundary. A closed surface S has no boundary like when we were proving Gauss's divergence theorem previously.

[3]Sometimes called a rotation operator.

Next, we sum the above equation over i or over all the small rects to arrive at

$$\sum_i \oint_{\Delta C_i} \mathbf{E}_i \cdot d\mathbf{l}_i = \sum_i \nabla \times \mathbf{E}_i \cdot \Delta \mathbf{S}_i \tag{2.2.4}$$

Again, on the left-hand side of the above, all the contour integrals over the small rects cancel each other internal to S save for those on the boundary. In the limit when $\Delta S_i \to 0$, the left-hand side becomes a contour integral over the larger contour C, and the right-hand side becomes a surface integral over S. One arrives at Stokes's theorem, which is

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} \tag{2.2.5}$$
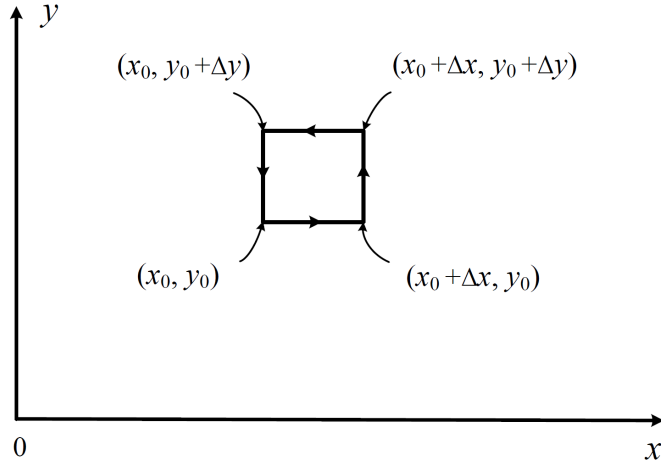

Figure 2.6: We approximate the integration over a small rect using this figure. There are four edges to this small rect.

Next, we need to prove the details of definition (2.2.2). Performing the integral over the small rect, one gets

$$\oint_{\Delta C} \mathbf{E} \cdot d\mathbf{l} = E_x(x_0, y_0, z_0)\Delta x + E_y(x_0 + \Delta x, y_0, z_0)\Delta y$$
$$- E_x(x_0, y_0 + \Delta y, z_0)\Delta x - E_y(x_0, y_0, z_0)\Delta y$$
$$= \Delta x \Delta y \left(\frac{E_x(x_0, y_0, z_0)}{\Delta y} - \frac{E_x(x_0, y_0 + \Delta y, z_0)}{\Delta y} \right.$$
$$\left. - \frac{E_y(x_0, y_0, z_0)}{\Delta x} + \frac{E_y(x_0, y_0 + \Delta y, z_0)}{\Delta x} \right)$$

$$\tag{2.2.6}$$

We have picked the normal to the incremental surface ΔS to be $\hat{z}$ in the above example, and hence, the above gives rise to the identity that

$$\lim_{\Delta S \to 0} \frac{\oint_{\Delta S} \mathbf{E} \cdot d\mathbf{l}}{\Delta S} = \frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x = \hat{z} \cdot \nabla \times \mathbf{E} \tag{2.2.7}$$

Picking different $\Delta \mathbf{S}$ with different orientations and normals $\hat{n}$, one gets

$$\frac{\partial}{\partial y} E_z - \frac{\partial}{\partial z} E_y = \hat{x} \cdot \nabla \times \mathbf{E} \tag{2.2.8}$$

$$\frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z = \hat{y} \cdot \nabla \times \mathbf{E} \tag{2.2.9}$$

Consequently, one gets

$$\nabla \times \mathbf{E} = \hat{x} \left(\frac{\partial}{\partial y} E_z - \frac{\partial}{\partial z} E_y \right) + \hat{y} \left(\frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z \right)$$
$$+ \hat{z} \left(\frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x \right) \tag{2.2.10}$$

where

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \tag{2.2.11}$$

2.2.1 Faraday's Law in Differential Operator Form

Faraday's law is experimentally motivated. Michael Faraday (1791-1867) was an extraordinary experimentalist who documented this law with meticulous care. It was only decades later that a mathematical description of this law was arrived at.

Faraday's law in integral form is given by[4]

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S} \tag{2.2.12}$$

Assuming that the surface S is not time varying, one can take the time derivative into the integrand and write the above as

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\iint_S \frac{\partial}{\partial t} \mathbf{B} \cdot d\mathbf{S} \tag{2.2.13}$$

One can replace the left-hand side with the use of Stokes' theorem to arrive at

$$\iint_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\iint_S \frac{\partial}{\partial t} \mathbf{B} \cdot d\mathbf{S} \tag{2.2.14}$$

---

[4]Faraday's law is experimentally motivated. Michael Faraday (1791-1867) was an extraordinary experimentalist who documented this law with meticulous care. It was only decades later that a mathematical description of this law was arrived at.

The normal of the surface element $d\mathbf{S}$ can be pointing in an arbitrary direction, and the surface S can be very small. Then the integral can be removed, and one has

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \qquad (2.2.15)$$

The above is Faraday's law in differential operator form.

In the static limit is

$$\nabla \times \mathbf{E} = 0 \qquad (2.2.16)$$

2.2.2 Physical Meaning of Curl Operator

The curl operator $\nabla\times$ is a measure of the rotation or the circulation of a field at a point in space. On the other hand, $\oint_{\Delta C} \mathbf{E} \cdot d\mathbf{l}$ is a measure of the circulation of the field $\mathbf{E}$ around the loop formed by C. Again, the curl operator has its complicated representations in other coordinate systems, a subject that will not be discussed in detail here.

It is to be noted that our proof of the Stokes's theorem is for a flat open surface S, and not for a general curved open surface. Since all curved surfaces can be tessellated into a union of flat triangular surfaces according to the tiling theorem, the generalization of the above proof to curved surface is straightforward. An example of such a triangulation of a curved surface into a union of triangular surfaces is shown in Figure 2.7.
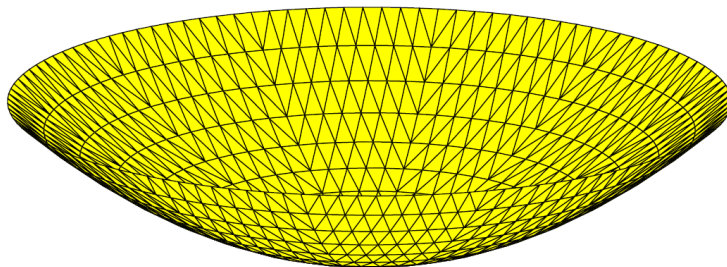

Figure 2.7: An arbitrary curved surface can be triangulated with flat triangular patches. The triangulation can be made arbitrarily accurate by making the patches arbitrarily small.

2.2.3 Example

Suppose $\mathbf{E} = \hat{\mathbf{x}}3y + \hat{\mathbf{y}}x$, calculate $\int \mathbf{E} \cdot d\mathbf{l}$ along a straight line in the x-y plane joining (0,0) to (3,1).

2.3　Maxwell's Equations in Differential Operator Form

With the use of Gauss' divergence theorem and Stokes' theorem, Maxwell's equations can be written more elegantly in differential operator forms. They are:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.3.1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{2.3.2}$$

$$\nabla \cdot \mathbf{D} = \varrho \tag{2.3.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.3.4}$$

These equations are point-wise relations as they relate field values at a given point in space. Moreover, they are not independent of each other. For instance, one can take the divergence of the first equation (2.3.1), making use of the vector identity that $\nabla \cdot \nabla \times \mathbf{E} = 0$, one gets

$$-\frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = 0 \rightarrow \nabla \cdot \mathbf{B} = \text{constant} \tag{2.3.5}$$

This constant corresponds to magnetic charges, and since they have not been experimentally observed, one can set the constant to zero. Thus the fourth of Maxwell's equations, (2.3.4), follows from the first (2.3.1).

Similarly, by taking the divergence of the second equation (2.3.2), and making use of the current continuity equation that

$$\nabla \cdot \mathbf{J} + \frac{\partial \varrho}{\partial t} = 0 \tag{2.3.6}$$

one can obtain the second last equation (2.3.3). Notice that in (2.3.3), the charge density ϱ can be time-varying, whereas in the previous lecture, we have "derived" this equation from Coulomb's law using electrostatic theory.

The above logic follows if $\partial/\partial t \neq 0$, and is not valid for static case. In other words, for statics, the third and the fourth equations are not derivable from the first two. Hence all four Maxwell's equations are needed for static problems. For electrodynamic problems, only solving the first two suffices.

Something is amiss in the above. If $\mathbf{J}$ is known, then solving the first two equations implies solving for four vector unknowns, $\mathbf{E}, \mathbf{H}, \mathbf{B}, \mathbf{D}$, which has 12 scalar unknowns. But there are only two vector equations or 6 scalar equations in the first two equations. Thus we need more equations. These are provide by the constitutive relations that we shall discuss next.

Bibliography

[1] J. A. Kong, "Theory of electromagnetic waves," *New York, Wiley-Interscience, 1975. 348 p.*, 1975.

[2] A. Einstein *et al.*, "On the electrodynamics of moving bodies," *Annalen der Physik*, vol. 17, no. 891, p. 50, 1905.

[3] P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation," *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 114, no. 767, pp. 243–265, 1927.

[4] R. J. Glauber, "Coherent and incoherent states of the radiation field," *Physical Review*, vol. 131, no. 6, p. 2766, 1963.

[5] C.-N. Yang and R. L. Mills, "Conservation of isotopic spin and isotopic gauge invariance," *Physical review*, vol. 96, no. 1, p. 191, 1954.

[6] G. t'Hooft, *50 years of Yang-Mills theory.* World Scientific, 2005.

[7] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* Princeton University Press, 2017.

[8] F. Teixeira and W. C. Chew, "Differential forms, metrics, and the reflectionless absorption of electromagnetic waves," *Journal of Electromagnetic Waves and Applications*, vol. 13, no. 5, pp. 665–686, 1999.

[9] W. C. Chew, E. Michielssen, J.-M. Jin, and J. Song, *Fast and efficient algorithms in computational electromagnetics.* Artech House, Inc., 2001.

[10] A. Volta, "On the electricity excited by the mere contact of conducting substances of different kinds. in a letter from Mr. Alexander Volta, FRS Professor of Natural Philosophy in the University of Pavia, to the Rt. Hon. Sir Joseph Banks, Bart. KBPR S," *Philosophical transactions of the Royal Society of London*, no. 90, pp. 403–431, 1800.

[11] A.-M. Ampère, *Exposé méthodique des phénomènes électro-dynamiques, et des lois de ces phénomènes.* Bachelier, 1823.

[12] ——, *Mémoire sur la théorie mathématique des phénomènes électro-dynamiques unique-ment déduite de l'expérience: dans lequel se trouvent réunis les Mémoires que M. Ampère a communiqués à l'Académie royale des Sciences, dans les séances des 4 et 26 décembre 1820, 10 juin 1822, 22 décembre 1823, 12 septembre et 21 novembre 1825.* Bachelier, 1825.

[13] B. Jones and M. Faraday, *The life and letters of Faraday.* Cambridge University Press, 2010, vol. 2.

[14] G. Kirchhoff, "Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird," *Annalen der Physik*, vol. 148, no. 12, pp. 497–508, 1847.

[15] L. Weinberg, "Kirchhoff's' third and fourth laws'," *IRE Transactions on Circuit Theory*, vol. 5, no. 1, pp. 8–30, 1958.

[16] T. Standage, *The Victorian Internet: The remarkable story of the telegraph and the nineteenth century's online pioneers.* Phoenix, 1998.

[17] J. C. Maxwell, "A dynamical theory of the electromagnetic field," *Philosophical trans-actions of the Royal Society of London*, no. 155, pp. 459–512, 1865.

[18] H. Hertz, "On the finite velocity of propagation of electromagnetic actions," *Electric Waves*, vol. 110, 1888.

[19] M. Romer and I. B. Cohen, "Roemer and the first determination of the velocity of light (1676)," *Isis*, vol. 31, no. 2, pp. 327–379, 1940.

[20] A. Arons and M. Peppard, "Einstein's proposal of the photon concept–a translation of the Annalen der Physik paper of 1905," *American Journal of Physics*, vol. 33, no. 5, pp. 367–374, 1965.

[21] A. Pais, "Einstein and the quantum theory," *Reviews of Modern Physics*, vol. 51, no. 4, p. 863, 1979.

[22] M. Planck, "On the law of distribution of energy in the normal spectrum," *Annalen der physik*, vol. 4, no. 553, p. 1, 1901.

[23] Z. Peng, S. De Graaf, J. Tsai, and O. Astafiev, "Tuneable on-demand single-photon source in the microwave range," *Nature communications*, vol. 7, p. 12588, 2016.

[24] B. D. Gates, Q. Xu, M. Stewart, D. Ryan, C. G. Willson, and G. M. Whitesides, "New approaches to nanofabrication: molding, printing, and other techniques," *Chemical re-views*, vol. 105, no. 4, pp. 1171–1196, 2005.

[25] J. S. Bell, "The debate on the significance of his contributions to the foundations of quantum mechanics, Bells Theorem and the Foundations of Modern Physics (A. van der Merwe, F. Selleri, and G. Tarozzi, eds.)," 1992.

[26] D. J. Griffiths and D. F. Schroeter, *Introduction to quantum mechanics.* Cambridge University Press, 2018.

[27] C. Pickover, *Archimedes to Hawking: Laws of science and the great minds behind them.* Oxford University Press, 2008.

[28] R. Resnick, J. Walker, and D. Halliday, *Fundamentals of physics.* John Wiley, 1988.

[29] S. Ramo, J. R. Whinnery, and T. Duzer van, *Fields and waves in communication electronics, Third Edition.* John Wiley & Sons, Inc., 1995.

[30] J. L. De Lagrange, "Recherches d'arithmétique," *Nouveaux Mémoires de l'Académie de Berlin*, 1773.

[31] J. A. Kong, *Electromagnetic Wave Theory.* EMW Publishing, 2008.

[32] H. M. Schey and H. M. Schey, *Div, grad, curl, and all that: an informal text on vector calculus.* WW Norton New York, 2005.

[33] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on physics, Vol. I: The new millennium edition: mainly mechanics, radiation, and heat.* Basic books, 2011, vol. 1.

[34] W. C. Chew, *Waves and fields in inhomogeneous media.* IEEE press, 1995.

[35] V. J. Katz, "The history of Stokes' theorem," *Mathematics Magazine*, vol. 52, no. 3, pp. 146–156, 1979.

[36] W. K. Panofsky and M. Phillips, *Classical electricity and magnetism.* Courier Corporation, 2005.

[37] T. Lancaster and S. J. Blundell, *Quantum field theory for the gifted amateur.* OUP Oxford, 2014.

[38] W. C. Chew, "Ece 350x lecture notes," *http://wcchew.ece.illinois.edu/chew/ece350.html*, 1990.

[39] C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory.* Springer Science & Business Media, 2013.

[40] J. M. Crowley, *Fundamentals of applied electrostatics.* Krieger Publishing Company, 1986.

[41] C. Balanis, *Advanced Engineering Electromagnetics.* Hoboken, NJ, USA: Wiley, 2012.

[42] J. D. Jackson, *Classical electrodynamics.* AAPT, 1999.

[43] R. Courant and D. Hilbert, *Methods of Mathematical Physics: Partial Differential Equations.* John Wiley & Sons, 2008.

[44] L. Esaki and R. Tsu, "Superlattice and negative differential conductivity in semiconductors," *IBM Journal of Research and Development*, vol. 14, no. 1, pp. 61–65, 1970.

[45] E. Kudeki and D. C. Munson, *Analog Signals and Systems.*   Upper Saddle River, NJ, USA: Pearson Prentice Hall, 2009.

[46] A. V. Oppenheim and R. W. Schafer, *Discrete-time signal processing.*   Pearson Education, 2014.

[47] R. F. Harrington, *Time-harmonic electromagnetic fields.*   McGraw-Hill, 1961.

[48] E. C. Jordan and K. G. Balmain, *Electromagnetic waves and radiating systems.*   Prentice-Hall, 1968.

[49] G. Agarwal, D. Pattanayak, and E. Wolf, "Electromagnetic fields in spatially dispersive media," *Physical Review B*, vol. 10, no. 4, p. 1447, 1974.

[50] S. L. Chuang, *Physics of photonic devices.*   John Wiley & Sons, 2012, vol. 80.

[51] B. E. Saleh and M. C. Teich, *Fundamentals of photonics.*   John Wiley & Sons, 2019.

[52] M. Born and E. Wolf, *Principles of optics: electromagnetic theory of propagation, interference and diffraction of light.*   Elsevier, 2013.

[53] R. W. Boyd, *Nonlinear optics.*   Elsevier, 2003.

[54] Y.-R. Shen, "The principles of nonlinear optics," *New York, Wiley-Interscience, 1984, 575 p.*, 1984.

[55] N. Bloembergen, *Nonlinear optics.*   World Scientific, 1996.

[56] P. C. Krause, O. Wasynczuk, and S. D. Sudhoff, *Analysis of electric machinery.*   McGraw-Hill New York, 1986, vol. 564.

[57] A. E. Fitzgerald, C. Kingsley, S. D. Umans, and B. James, *Electric machinery.*   McGraw-Hill New York, 2003, vol. 5.

[58] M. A. Brown and R. C. Semelka, *MRI.: Basic Principles and Applications.*   John Wiley & Sons, 2011.

[59] C. A. Balanis, *Advanced engineering electromagnetics.*   John Wiley & Sons, 1999.

[60] Wikipedia, "Lorentz force," 2019.

[61] R. O. Dendy, *Plasma physics: an introductory course.*   Cambridge University Press, 1995.

[62] P. Sen and W. C. Chew, "The frequency dependent dielectric and conductivity response of sedimentary rocks," *Journal of microwave power*, vol. 18, no. 1, pp. 95–105, 1983.

[63] D. A. Miller, *Quantum Mechanics for Scientists and Engineers.*   Cambridge, UK: Cambridge University Press, 2008.

[64] W. C. Chew, "Quantum mechanics made simple: Lecture notes," *http://wcchew.ece.illinois.edu/chew/course/QMAll20161206.pdf*, 2016.

[65] B. G. Streetman, S. Banerjee *et al.*, *Solid state electronic devices.* Prentice hall Englewood Cliffs, NJ, 1995, vol. 4.

[66] Smithsonian, https://www.smithsonianmag.com/history/this-1600-year-old-goblet-shows-that-the-romans-were-nanotechnology-pioneers-787224/, accessed: 2019-09-06.